On one connection between moments of random variables

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TBILISI
December, 18, 2023

This work is supported by the European Union's Horizon Europe Grant Project "GAIN"

1. Introduction

$(\Omega, \mathcal{A}, \mathbb{P})$ – a probability space; $\xi : \Omega \to \mathbb{R}^1$ – a random variable; $\mathbb{E}$ – mathematical expectation symbol.

Theorem 1.1

(R. Fukuda, 1990) Let $p > q > 0$ *and for some* $C \geq 1$*:*

$$\left\{\mathbb{E}|\xi|^p\right\}^{1/p} \leq C \left\{\mathbb{E}|\xi|^q\right\}^{1/q} < \infty.$$

Then for any r, s*,* $0 < r, s \leq p$ *there exists a constant* K *which depends only on* r, s, p, q *and* C *such that*

$$\left\{\mathbb{E}|\xi|^r\right\}^{1/r} \leq K \left\{\mathbb{E}|\xi|^s\right\}^{1/s}.$$

1. Introduction

$\xi : \Omega \to \mathbb{R}^1$ – a Gaussian random variable with $\mathbb{E}\,\xi = 0$;

$p > 0$ – a real number;

then

$$\left(\mathbb{E}|\xi|^p\right)^{1/p} = \frac{2^{1/2}}{\pi^{1/2p}} \Gamma^{1/p}\left(\frac{p+1}{2}\right)\left\{\mathbb{E}|\xi|^2\right\}^{1/2},$$

where for any positive number $x > 0$ Gamma function is defined as follows

$$\Gamma(x) = \int\limits_0^\infty t^{x-1} e^{-t}\, dt.$$

We have improved a little bit Fukuda's aforementioned result and formulate it as follows:

Theorem 2.1

Let $p > q > 0$ *and for some* $C \geq 1$*:*

$$\left\{\mathbb{E}|\xi|^p\right\}^{1/p} \leq C \left\{\mathbb{E}|\xi|^q\right\}^{1/q} < \infty.$$

Then for any r, s*,* $0 < r, s \leq p$*, we have*

$$\left\{\mathbb{E}|\xi|^r\right\}^{1/r} \leq C^{\beta} \left\{\mathbb{E}|\xi|^s\right\}^{1/s},$$

where

$$\beta = \begin{cases} 0, & \text{if } 0 < r \leq s \leq p, \\ 1, & \text{if } q \leq s < r \leq p, \\ \frac{q(p-s)}{s(p-q)}, & \text{if } 0 < s < q < r \leq p, \\ \frac{p(q-s)}{s(p-q)}, & \text{if } 0 < s < r \leq q. \end{cases}$$

3. Some Remarks

Note

1. This theorem refines Fukuda's constant.

2. The constant C^{β} is more exact and reflects the dependence on the parameters p, q, r, s and C explicitly.

2. Main Theorem

Theorem 2.1

Let $p > q > 0$ *and for some* $C \geq 1$*:*

$$\left\{\mathbb{E}|\xi|^p\right\}^{1/p} \leq C\left\{\mathbb{E}|\xi|^q\right\}^{1/q} < \infty.$$

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$$\left\{\mathbb{E}|\xi|^r\right\}^{1/r} \leq C^{\beta}\left\{\mathbb{E}|\xi|^s\right\}^{1/s},$$

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Since the expression $\{\mathbb{E}|\xi|^{\alpha}\}^{1/\alpha}$, as a function of α, $\alpha > 0$, is nondecreasing, the statement of the theorem for the case $0 < r \leq s \leq p$ is evident.

For the case $q \leq s < r \leq p$, the proof is also easy using the condition of the theorem in addition.

Therefore, we begin the proof with the case $0 < s < q < r \leq p$.

Introduce the numbers $u = \frac{p(q-s)}{q(p-s)}$ and $v = \frac{s(p-q)}{q(p-s)}$.

Clearly $0 < u, v < 1$ and $u + v = 1$.

Using the Holder inequality we get the following inequality:

$$\mathbb{E}|\xi|^q = \mathbb{E}|\xi|^{(u+v)q} = \mathbb{E}|\xi|^{uq} \cdot |\xi|^{vq} \leq$$
$$\leq \left\{\mathbb{E}|\xi|^{uq\alpha}\right\}^{1/\alpha} \cdot \left\{\mathbb{E}|\xi|^{vq\alpha^*}\right\}^{1/\alpha^*}, \qquad (4.1)$$

where $1 < \alpha, \alpha^* < \infty$ and $1/\alpha + 1/\alpha^* = 1$.

Choose now α by the condition $uq\alpha = p$. It is clear that $\alpha > 1$ and

$$\alpha = \frac{p-s}{q-s}, \qquad \alpha^* = \frac{\alpha}{\alpha - 1} = \frac{p-s}{p-q}.$$

For such number α the relation (4.1) leads to the following one

4. Proof of Theorem 2.1 (continuation)

$$\mathbb{E}|\xi|^q \leq \left\{\mathbb{E}|\xi|^p\right\}^{\frac{q-s}{p-s}} \cdot \left\{\mathbb{E}|\xi|^s\right\}^{\frac{p-q}{p-s}} \leq$$

$$\leq C^{\frac{p(q-s)}{p-s}} \left\{\mathbb{E}|\xi|^q\right\}^{\frac{p(q-s)}{q(p-s)}} \cdot \left\{\mathbb{E}|\xi|^s\right\}^{\frac{p-q}{p-s}},$$

from which it can be easily obtained the inequality which is the key point for our proof

$$\left\{\mathbb{E}|\xi|^q\right\}^{1/q} \leq C^{\frac{p(q-s)}{s(p-q)}} \left\{\mathbb{E}|\xi|^s\right\}^{1/s}. \qquad (4.2)$$

4. Proof of Theorem 2.1 (continuation)

Using now Holder inequality, the assumption of the theorem and finally the key inequality (4.2) we get:

$$\left\{\mathbb{E}|\xi|^r\right\}^{1/r} \le \left\{\mathbb{E}|\xi|^p\right\}^{1/p} \le C\left\{\mathbb{E}|\xi|^q\right\}^{1/q} \le C^{\frac{q(q-s)}{s(p-q)}}\left\{\mathbb{E}|\xi|^s\right\}^{1/s}.$$

4. Proof of Theorem 2.1 (continuation)

Now the case $0 < s < r \leq q$ is left, which can be reduced to the previous one.

Indeed, by (4.2) we get the inequality:

$$\left\{\mathbb{E}|\xi|^r\right\}^{1/r} \leq \left\{\mathbb{E}|\xi|^q\right\}^{1/q} \leq C^{\frac{p(q-s)}{s(p-q)}} \left\{\mathbb{E}|\xi|^s\right\}^{1/s}$$

which ends the proof.

2. Main Theorem

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Let $p > q > 0$ *and for some* $C \geq 1$*:*

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Then for any r, s*,* $0 < r, s \leq p$*, we have*

$$\left\{\mathbb{E}|\xi|^r\right\}^{1/r} \leq C^{\beta} \left\{\mathbb{E}|\xi|^s\right\}^{1/s},$$

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$$r = p, \qquad s = 1.$$

The Fukuda constant:

$$C^{1+\frac{pq}{p-q}} \cdot q \cdot B^{-1}\left(\frac{1}{q}, \frac{p}{p-q} + 1\right)$$

where $B(x, y)$ is the beta function and is defined as follows

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad x, y > 0.$$

$$r = p, \qquad s = 1.$$

The constant of Theorem 2.1:

$$C^{\beta} = \begin{cases} C, & \text{for} \quad 0 < q \leq 1, \\ C^{\frac{q(p-1)}{p-q}}, & \text{for} \quad 1 < q < p. \end{cases}$$

4. Main Theorem (continuation)

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THANK YOU!